# STSM report

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## I. PURPOSE OF THE STSM

The aim of the STSM was to discuss the implementation and test the fCCZ4 formulation of the Einstein equations in the 3D numerical relativity code NADA, recently proposed in a recent paper [1]. This formulation is a generalization of a covariant and conformal version of the Z4 system of the Einstein equations by adopting a reference metric approach, well suited for curvilinear as well as Cartesian coordinates. It also includes a constraint damping scheme that allows to control the violation of the constraints by adding two damping parameters, $\kappa_1$ and $\kappa_2$.

## II. DESCRIPTION OF THE WORK CARRIED OUT DURING THE STSM

In this section, we present the evolution equations of the fCCZ4 formulation we have implemented in the code. Defining

$$\partial_\perp \equiv \partial_t - \mathcal{L}_\beta \tag{II.1}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative along the shift vector $\beta^i$, the fully covariant and conformal Z4 system in a reference-metric approach (fCCZ4) is then given by the following set of evolution equations:

$$\partial_\perp \bar{\gamma}_{ij} = -\frac{2}{3}\bar{\gamma}_{ij}\bar{\mathcal{D}}_k\beta^k - 2\alpha\bar{A}_{ij}, \tag{II.2}$$

$$\begin{aligned}\partial_\perp \bar{A}_{ij} = & -\frac{2}{3}\bar{A}_{ij}\bar{\mathcal{D}}_k\beta^k - 2\alpha\bar{A}_{ik}\bar{A}^k_j + \alpha\bar{A}_{ij}(K-2\Theta) + e^{-4\phi}\big[-2\alpha\bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\phi + 4\alpha\bar{\mathcal{D}}_i\phi\bar{\mathcal{D}}_j\phi + 4\bar{\mathcal{D}}_{(i}\alpha\bar{\mathcal{D}}_{j)}\phi - \bar{\mathcal{D}}_i\bar{\mathcal{D}}_j\alpha \\ & +\alpha(\bar{R}_{ij} + \mathcal{D}_i Z_j + \mathcal{D}_j Z_i - 8\pi S_{ij})\big]^{\mathrm{TF}},\end{aligned} \tag{II.3}$$

$$\partial_\perp \phi = \frac{1}{6}\bar{\mathcal{D}}_i\beta^i - \frac{1}{6}\alpha K, \tag{II.4}$$

$$\begin{aligned}\partial_\perp K = & \; e^{-4\phi}\big[\alpha\big(\bar{R} - 8\bar{\mathcal{D}}^i\phi\bar{\mathcal{D}}_i\phi - 8\bar{\mathcal{D}}^2\phi\big) - \big(2\bar{\mathcal{D}}^i\alpha\bar{\mathcal{D}}_i\phi + \bar{\mathcal{D}}^2\alpha\big)\big] + \alpha(K^2 - 2\Theta K) + 2\alpha\mathcal{D}_i Z^i - 3\alpha\kappa_1(1+\kappa_2)\Theta \\ & +4\pi\alpha(S - 3E),\end{aligned} \tag{II.5}$$

$$\begin{aligned}\partial_\perp \Theta = & \; \frac{1}{2}\alpha\big[e^{-4\phi}\big(\bar{R} - 8\bar{\mathcal{D}}^i\phi\bar{\mathcal{D}}_i\phi - 8\bar{\mathcal{D}}^2\phi\big) - \bar{A}^{ij}\bar{A}_{ij} + \frac{2}{3}K^2 - 2\Theta K + 2\mathcal{D}_i Z^i\big] - Z^i\partial_i\alpha - \alpha\kappa_1(2+\kappa_2)\Theta \\ & -8\pi\alpha E,\end{aligned} \tag{II.6}$$

$$\begin{aligned}\partial_\perp \tilde{\Lambda}^i = & \; \bar{\gamma}^{jk}\hat{\mathcal{D}}_j\hat{\mathcal{D}}_k\beta^i + \frac{2}{3}\Delta\Gamma^i\bar{\mathcal{D}}_j\beta^j + \frac{1}{3}\bar{\mathcal{D}}^i\bar{\mathcal{D}}_j\beta^j - 2\bar{A}^{jk}\big(\delta^i{}_j\partial_k\alpha - 6\alpha\delta^i{}_j\partial_k\phi - \alpha\Delta\Gamma^i_{jk}\big) - \frac{4}{3}\alpha\bar{\gamma}^{ij}\partial_j K \\ & +2\bar{\gamma}^{ki}\big(\alpha\partial_k\Theta - \Theta\partial_k\alpha - \frac{2}{3}\alpha K Z_k\big) - 2\alpha\kappa_1\bar{\gamma}^{ij}Z_j - 16\pi\alpha\bar{\gamma}^{ij}S_j.\end{aligned} \tag{II.7}$$

Here the superscript TF denotes the trace-free part of a tensor, $\kappa_1$ and $\kappa_2$ are the damping coefficients introduced by [**?** ], and $\hat{\mathcal{D}}_i$, $\mathcal{D}_i$ and $\bar{\mathcal{D}}_i$ denote the covariant derivatives built from the connection associated with the reference metric $\hat{\gamma}_{ij}$, the physical metric $\gamma_{ij}$ and the conformal metric $\bar{\gamma}_{ij}$, respectively. We have also defined

$$\tilde{\Lambda}^i \equiv \bar{\Lambda}^i + 2\bar{\gamma}^{ij}Z_j, \tag{II.8}$$

where

$$\bar{\Lambda}^i \equiv \Delta\Gamma^i = \bar{\gamma}^{jk}\Delta\Gamma^i_{jk}. \tag{II.9}$$

The vector $\tilde{\Lambda}^i$ plays the role of the "conformal connection functions" in the original CCZ4 system; its evolution equation (II.7) involves the evolution equation for the variables $Z_i$.

The matter sources $E$, $S_i$, $S_{ij}$ and $S$ denote the density, momentum density, stress, and the trace of the stress as observed by a normal observer, respectively:

$$E \equiv n_\mu n_\nu T^{\mu\nu}, \tag{II.10}$$

$$S_i \equiv -\gamma_{i\mu} n_\nu T^{\mu\nu}, \tag{II.11}$$

$$S_{ij} \equiv \gamma_{i\mu}\gamma_{j\nu} T^{\mu\nu}, \tag{II.12}$$

$$S \equiv \gamma^{ij} S_{ij}. \tag{II.13}$$

In Eq. (II.3), we compute the Ricci tensor $\bar{R}_{ij}$ associated with $\bar{\gamma}_{ij}$ from

$$\bar{R}_{ij} = -\frac{1}{2}\bar{\gamma}^{kl}\hat{\mathcal{D}}_k\hat{\mathcal{D}}_l\bar{\gamma}_{ij} + \bar{\gamma}_{(i}\hat{\mathcal{D}}_{j)}\Delta\Gamma^k + \Delta\Gamma^k\Delta\Gamma_{(ij)k} + \bar{\gamma}^{kl}\big(2\Delta\Gamma^m_{k(i)}\Delta\Gamma_{j)ml} + \Delta\Gamma^m_{ik}\Delta\Gamma_{mjl}\big). \tag{II.14}$$

Here we compute the $\Delta\Gamma^i$ from their definition (II.9). Given $\Delta\Gamma^i$, and values for $\tilde{\Lambda}^i$, the vectors $Z_i$, which are not evolved independently, can be determined from (II.8).

Unless stated otherwise we fix the gauge freedom by imposing the so called “non-advective 1+log” condition for the lapse

$$\partial_t\alpha = -2\alpha(K - 2\Theta), \tag{II.15}$$

and a variation of the ”Gamma-driver” condition for the shift vector

$$\partial_t\beta = B^i, \tag{II.16}$$

$$\partial_t B^i = \frac{3}{4}\partial_t\tilde{\Lambda}^i. \tag{II.17}$$

Finally, when $\Theta = Z_i = 0$, the evolution equations (II.2)-(II.7) imply that the Hamiltonian and momentum constraints hold in the form

$$\mathcal{H} \equiv \frac{2}{3}K^2 - \bar{A}_{ij}\bar{A}^{ij} + e^{-4\phi}\big(\bar{R} - 8\bar{\mathcal{D}}^i\phi\bar{\mathcal{D}}_i\phi - 8\bar{\mathcal{D}}^2\phi\big) - 16\pi E = 0, \tag{II.18}$$

$$\mathcal{M}^i \equiv e^{-4\phi}\big(\frac{1}{\sqrt{\bar{\gamma}}}\hat{\mathcal{D}}_j(\sqrt{\bar{\gamma}}\bar{A}^{ij}) + 6\bar{A}^{ij}\partial_j\phi - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K + \bar{A}^{jk}\Delta\Gamma^i_{jk}\big) - 8\pi S^i = 0, \tag{II.19}$$

where $\bar{R}$ is the trace of $\bar{R}_{ij}$.

We discussed the main features of the implementation the fCCZ4 formulation in the fully 3D code NADA [2], able to perform numerical evolutions using spherical polar coordinates without any symmetry assumption adopting a covariant form of the BSSN formulation. The fCCZ4 formulation have been tested by performing two tests in 3D and comparing our results with those obtained using the BSSN formulation. We expected to find the same trend of the spherically symmetric case and to observe a reduction in the violations of the Hamiltonian constraint of several orders of magnitude for fCCZ4 with respect to BSSN. However, at the beginning, we found what seemed to be a numerical instability at the origin that after some interations crashed the simulation. So we were not able to obtain stable evolutions initially.

We investigated this behaviour and discovered that some ’stiff’ terms seemed to be causing the instabilities. They were related to the three vector $Z_i$ and to the connection functions $\Delta\Gamma^i$, so we worked out an extrapolation at the origin for those quantities, obtaining stable numerical evolutions for the tests we performed.

## III. DESCRIPTION OF THE MAIN RESULTS OBTAINED

We performed two tests, namely the propagation of an axisymmetric Teukolsky wave and the evolution of a rotating neutron star in equilibrium, in order to compare the Hamiltonian constraint for fCCZ4 with respect to BSSN.

### A. Weak gravitational waves

As a first test of our codes we consider small-amplitude gravitational waves on a flat Minkowski background. We construct an analytical, linear solution for quadrupolar ($l$=2) waves from a function

$$F(r,t) = A(r \mp t)e^{-(r\mp t)^2/\lambda^2} \tag{III.1}$$

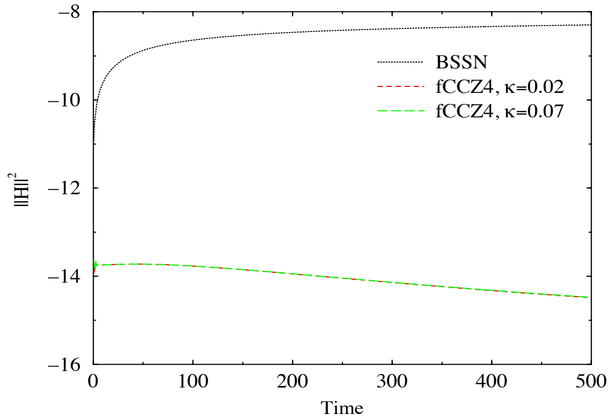


FIG. 1: Time evolution of the L2 norm of the Hamiltonian constraint of the Teukolsky wave for BSSN and fCCZ4 for different choices of the damping parameter $\kappa_1$, with $\kappa_2 = 0$.

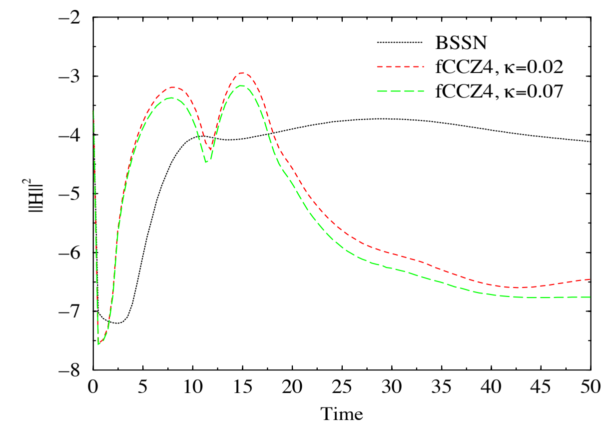


FIG. 2: Time evolution of the L2 norm of the Hamiltonian constraint of the rotating neutron star for BSSN and fCCZ4 for different choices of the damping parameter $\kappa_1$, with $\kappa_2 = 0$.

where the constant $A$ is related to the amplitude of the wave and $\lambda$ to its wavelength, and we consider the axisymmetric mode $m = 0$. We set $\lambda = 1$ and choose and small amplitud $A = 10{-}7$ so we are in the linear regime.

In fig. (1), we show the L2-norm of the Hamiltonian constraint for BSSN and fCCZ4 for two values of the damping parameter $\kappa_1$. In [1], we realized that the influence of $\kappa_2$ is not important and can even lead to over-damping effects, so we kept $\kappa_2$ equal to zero. The L2-norm of the Hamiltonian constraint is almost five orders of magnitude smaller for fCCZ4 than for BSSN, as expected. The difference between the two choice of the damping parameters is not noticeable in this case.

### B. Rotating neutron star

Next, we evolved a rapidly rotating neutron star in equilibrium, the initial data constructed using the RNS code [4], solving the coupled system formed by the Einstein equations and the general relativistic hydrodinamics equations. In particular, we chose the uniformly rotating BU7 model from [5], a relativistic polytrope taking the polytropic index $N = 1$ and the polytropic constant $K = 100$. This model has a central energy density $\epsilon_c = 1.444 \times 10^{-3}$ and the ratio of polar to equatorial coordinate axis being $r_p/r_e = 0.65$, resulting in a star with mass $M = 1.666$, radius $R = 12.3$.

In fig. 2, we plot the L2-norm of the Hamiltonian constraint for BSSN and fCCZ4 for the same choices of the damping parameter $\kappa_1$ and $\kappa_2$. Again, we observe a reduction in the violation of the Hamiltonian constraint up to almost three orders of magnitude with respect to BSSN. Besides, taking a large value of $\kappa_1$ leads to smaller constrain violations.

## IV. AFTER THE STSM

I plan to continue collaborating with Dr. Pedro Montero, using his 3D numerical relativity code NADA. We want to perform more demanding simulations, such as Kerr Black Hole and collapse of rotating neutron stars to black hole, in order to test the fCCZ4 formulation, comparing the results with those of the BSSN formulation, and study the influence of the damping parameters in those tests. Long term simulations of the Teukolsky waves and the rotating neutron star also have to be carried out.

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